

March 2000

Comments on  $N = 4$  Superconformal AlgebrasJørgen Rasmussen<sup>1</sup>*Physics Department, University of Lethbridge  
Lethbridge, Alberta, Canada T1K 3M4***Abstract**

We present a new and asymmetric  $N = 4$  superconformal algebra for arbitrary central charge, thus completing our recent work on its classical analogue with vanishing central charge. Besides the Virasoro generator and 4 supercurrents, the algebra consists of an internal  $SL(2) \otimes U(1)$  Kac-Moody algebra in addition to two spin 1/2 fermions and a bosonic scalar. The algebra is shown to be invariant under a linear twist of the generators, except for a unique value of the continuous twist parameter. At this value, the invariance is broken and the algebra collapses to the small  $N = 4$  superconformal algebra. In the context of string theory, the asymmetric  $N = 4$  superconformal algebra is provided with an explicit construction on the boundary of  $AdS_3$ , and is induced by an affine  $SL(2|2)$  current superalgebra residing on the world sheet. Substituting the world sheet  $SL(2|2)$  by the coset  $SL(2|2)/U(1)$  results in the small  $N = 4$  superconformal algebra on the boundary of  $AdS_3$ .

*PACS:* 11.25.Hf; 11.25.-w*Keywords:* Superconformal field theory;  $AdS/CFT$  correspondence; string theory; free field realization

---

<sup>1</sup>e-mail address: rasmussj@cs.uleth.ca

# 1 Introduction

Motivated by the exciting progress on the correspondence between string theory on anti-de Sitter space ( $AdS$ ) and conformal field theory [1, 2, 3], we have recently outlined an explicit construction of an infinite dimensional class of superconformal algebras (SCAs) on the boundary of  $AdS_3$  [4]. These space-time algebras are  $N$  extended SCAs induced by an affine  $SL(2|N/2)$  current superalgebra residing on the world sheet ( $N$  is even). Our construction generalizes a work by Ito [5] in which  $N = 1, 2$  and 4 SCAs are studied. Our constructions are both supersymmetric extensions of the Giveon-Kutasov-Seiberg construction of the Virasoro algebra [6]. A related approach to building  $N = 1, 2$  and 4 SCAs for fixed central charges may be found in Ref. [7].

In the present paper we shall complete our construction of the new and *asymmetric*  $N = 4$  SCA discovered in Ref. [4], where the case of vanishing central charge was treated. Here we shall provide the algebra for generic central charge. Besides the Virasoro generator and 4 supercurrents, the algebra consists of an internal  $SL(2) \otimes U(1)$  Kac-Moody algebra in addition to two spin 1/2 fermions and a bosonic scalar. Hence, it is not included in the standard classification of  $N = 4$  SCAs [8, 9, 10, 11, 12].

We shall also show that replacing the world sheet  $SL(2|2)$  current superalgebra by that of the coset  $SL(2|2)/U(1)$  induces the well known small  $N = 4$  SCA rather than the bigger, asymmetric  $N = 4$  SCA induced by  $SL(2|2)$ . We hope this will clarify the incompatibility between our work and the result for  $N = 4$  in Ref. [5]; there a construction (similar to ours) based on  $SL(2|2)$  was announced to result in the small  $N = 4$  SCA.

The new and asymmetric  $N = 4$  SCA is invariant under a one-parameter twist. The starting point for this observation is a linear modification of the Giveon-Kutasov-Seiberg Virasoro generator. The remaining twisted generators are obtained by requiring the SCA to be invariant. All twisted generators are given in terms of linear combinations of the original untwisted generators. It is an interesting observation that for precisely one value of the continuous twist parameter, the invariance is broken and the twisting results in the small  $N = 4$  SCA. It is stressed that the invariance and this collapse of the asymmetric SCA to the small  $N = 4$  SCA, are both independent of our construction and rely solely on the defining (anti-)commutators of the two SCAs.

The rest of this paper is organized as follows. In Section 2 we summarize briefly some of our results [4] on the general construction of SCAs induced by affine  $SL(2|N/2)$  current superalgebras.

Section 3 is devoted to  $N = 4$  SCAs, where the new and asymmetric algebra is provided. The small  $N = 4$  is shown to be induced by the coset  $SL(2|2)/U(1)$  current superalgebra, and the invariance of the asymmetric SCA is discussed.

Section 4 contains concluding remarks, while details on the Lie superalgebra  $sl(2|M)$  along with explicit free field realizations of the currents, are given in appendices A, B and C.

## 2 $N$ Extended Superconformal Algebras

## 2.1 Free Field Realizations of Affine Current Superalgebras

Associated to a Lie superalgebra  $\mathfrak{g}$  is an affine Lie superalgebra characterized by the central extension  $k$ . Associated to an affine Lie superalgebra is an affine current superalgebra whose generators,  $J_a$ , are conformal spin one primary fields (with respect to the Sugawara construction) and have the mutual operator product expansions (OPEs)

$$J_a(z)J_b(w) = \frac{\kappa_{a,b}k}{(z-w)^2} + \frac{f_{a,b}{}^c J_c(w)}{z-w} \quad (1)$$

$\kappa_{a,b}$  and  $f_{a,b}{}^c$  are the Cartan-Killing form and structure constants, respectively, of the underlying Lie superalgebra. Regular terms have been omitted. The general free field realization obtained in Ref. [13] is based on a pair of free ghost fields  $(\beta_\alpha, \gamma^\alpha)$  of conformal weights  $(1,0)$  for every positive root  $\alpha \in \Delta_+$  (written  $\alpha > 0$ ), and on a free scalar boson  $\varphi_i$  for every Cartan index  $i = 1, \dots, r$ , where  $r$  is the rank of the underlying Lie superalgebra. They satisfy the OPEs

$$\beta_\alpha(z)\gamma^{\alpha'}(w) = \frac{\delta_\alpha^{\alpha'}}{z-w}, \quad \varphi_i(z)\varphi_j(w) = \kappa_{i,j} \ln(z-w) \quad (2)$$

Note that in this notation, the ghost fields  $(\beta_\alpha, \gamma^\alpha)$  associated to *odd* roots  $\alpha \in \Delta_+^1$  are *fermionic*.  $(\Delta_\pm^1)$   $\Delta_\pm^0$  denotes the space of positive or negative (odd) even roots of the underlying Lie superalgebra, and  $\Delta_\pm = \Delta_\pm^0 \cup \Delta_\pm^1$ . The corresponding energy-momentum tensor is

$$T = \sum_{\alpha > 0} \partial\gamma^\alpha \beta_\alpha + \frac{1}{2} \partial\varphi \cdot \partial\varphi - \frac{1}{\sqrt{k+h^\vee}} \rho \cdot \partial^2\varphi \quad (3)$$

$\rho$  and  $h^\vee$  are the Weyl vector and the dual Coxeter number, respectively, of the underlying Lie superalgebra. Normal ordering is implicit. The generators of the affine current superalgebra are realized according to [13]

$$J_a(z) = \sum_{\alpha > 0} V_a^\alpha(\gamma(z))\beta_\alpha(z) + \sqrt{k+h^\vee} \sum_{j=1}^r P_a^j(\gamma(z))\partial\varphi_j(z) + J_a^{\text{anom}}(\gamma(z), \partial\gamma(z)) \quad (4)$$

where

$$J_a^{\text{anom}}(\gamma(z), \partial\gamma(z)) = \begin{cases} 0 & \text{for } a = i, \alpha > 0 \\ \sum_{\alpha' > 0} \partial\gamma^{\alpha'}(z) F_{\alpha, \alpha'}(\gamma(z)) & \text{for } a = \alpha < 0 \end{cases} \quad (5)$$

The explicit form of  $F_{\alpha, \alpha'}$  is not needed here but may be found in refs. [13, 4]. For  $\alpha = \alpha_i$  a simple root  $F_{\alpha_i, \alpha'}$  is a constant independent of the ghost fields  $\gamma$ :

$$F_{\alpha_i, \alpha'}(\gamma) = \frac{1}{2} \delta_{\alpha_i, \alpha'} ((2k + h^\vee) \kappa_{\alpha_i, -\alpha_i} - A_{ii}) \quad (6)$$

$A_{ij}$  is the Cartan matrix of the underlying Lie superalgebra, and is related to the Cartan-Killing form as  $\kappa_{i,j} = A_{ij} \kappa_{\alpha_j, -\alpha_j}$ .  $V$  and  $P$  are given by

$$V_\alpha^{\alpha'}(\gamma) = [B(C(\gamma))]_\alpha^{\alpha'}$$

$$\begin{aligned}
V_i^{\alpha'}(\gamma) &= -[C(\gamma)]_i^{\alpha'} \\
V_{-\alpha}^{\alpha'}(\gamma) &= \sum_{\alpha'' > 0} [e^{-C(\gamma)}]_{-\alpha}^{\alpha''} [B(-C(\gamma))]_{\alpha''}^{\alpha'} \\
P_{\alpha}^j(\gamma) &= 0 \\
P_i^j(\gamma) &= \delta_i^j \\
P_{-\alpha}^j(\gamma) &= [e^{-C(\gamma)}]_{-\alpha}^j
\end{aligned} \tag{7}$$

$B(u)$  is the generating function for the Bernoulli numbers  $B_n$

$$B(u) = \frac{u}{e^u - 1} = \sum_{n \geq 0} \frac{B_n}{n!} u^n \tag{8}$$

whereas the matrix  $C$  is defined by

$$C_a^b(\gamma) = - \sum_{\alpha > 0} \gamma^\alpha f_{\alpha, a}^b \tag{9}$$

The formal power series (7) all truncate and are thus polynomials.

## 2.2 Superconformal Algebra Generators

Most Lie superalgebras with even subalgebra  $\mathfrak{g}^0 = sl(2) \oplus \mathfrak{g}'$  have the property that the embedding of  $sl(2)$  in  $\mathfrak{g}$  carried by the odd generators (the set of which is denoted  $\mathfrak{g}^1$ ) is a fundamental (spin 1/2) representation<sup>1</sup>. This means that the space of odd roots may be divided into two parts

$$\Delta^1 = \Delta^{1-} \cup \Delta^{1+} \tag{10}$$

The roots  $\alpha^\pm \in \Delta^{1\pm}$  are characterized by

$$\frac{\alpha_{sl(2)} \cdot \alpha^\pm}{\alpha_{sl(2)}^2} = \pm \frac{1}{2} \tag{11}$$

and we have the correspondence

$$\Delta^{1+} = \alpha_{sl(2)} + \Delta^{1-} \tag{12}$$

Here  $\alpha_{sl(2)}$  is the positive root associated to the embedded  $sl(2)$ . In particular, the Lie superalgebra  $sl(2|N/2)$  allows such a decomposition of the root space. In the distinguished representation of  $sl(2|N/2)$  discussed in Appendix A, the embedding is associated to the simple root  $\alpha_1$ , while the only odd simple root is  $\alpha_2$ . Furthermore, the root space enjoys the refined decomposition

$$\Delta_+^1 = \Delta_+^{1-} \cup \Delta_+^{1+} \tag{13}$$

This refinement is not valid for all Lie superalgebras respecting (10), as  $osp(1|2)$  illustrates. In the following we shall concentrate on SCAs induced by affine  $SL(2|N/2)$  current superalgebras.

---

<sup>1</sup>This is true for all basic Lie superalgebras with even subalgebra  $\mathfrak{g}^0 = sl(2) \oplus \mathfrak{g}'$  except  $osp(3|2M)$  where the embedding is a spin 1 representation, see e.g. [14].

Now, the Virasoro algebra is induced by the embedded  $SL(2)$  and is generated by

$$\begin{aligned} L_n &= \oint \frac{dz}{2\pi i} \mathcal{L}_n(z) \\ \mathcal{L}_n &= a_+(n) (\gamma^{\alpha_1})^{n+1} E_{\alpha_1} + a_3(n) (\gamma^{\alpha_1})^n H_1 + a_-(n) (\gamma^{\alpha_1})^{n-1} F_{\alpha_1} \end{aligned} \quad (14)$$

The central charge is

$$c = -6k_1^\vee p_1 \quad (15)$$

$p_1$  is the winding number

$$p_1 = \oint \frac{dz}{2\pi i} \frac{\partial \gamma^{\alpha_1}}{\gamma^{\alpha_1}} \quad (16)$$

while  $k_1^\vee = \kappa_{\alpha_1, -\alpha_1} k$  is the level of the embedded  $sl(2)$  or the level in the direction  $\alpha_1$ . The constants  $a_+$ ,  $a_3$  and  $a_-$  are defined by

$$a_+(n) = \frac{1}{2}(n - n^2), \quad a_3(n) = \frac{1}{2}(1 - n^2), \quad a_-(n) = \frac{1}{2}(n + n^2) \quad (17)$$

In Ref. [4] it was found that for each *pair* of roots  $(\alpha^-, \alpha^+)$  we have a pair of supercurrents of spin 3/2 with respect to (14):

$$\begin{aligned} G_{\alpha^-; n+1/2} &= \oint \frac{dz}{2\pi i} \mathcal{G}_{\alpha^-; n+1/2}(z) \\ \mathcal{G}_{\alpha^-; n+1/2} &= (n+1)(\gamma^{\alpha_1})^n E_{\alpha^-} - n(\gamma^{\alpha_1})^{n+1} E_{\alpha^+} \end{aligned} \quad (18)$$

and

$$\begin{aligned} \overline{G}_{-\alpha^-; n-1/2} &= \oint \frac{dz}{2\pi i} \overline{\mathcal{G}}_{-\alpha^-; n-1/2}(z) \\ \overline{\mathcal{G}}_{-\alpha^-; n-1/2} &= (n-1)(\gamma^{\alpha_1})^n F_{\alpha^-} + n(\gamma^{\alpha_1})^{n-1} F_{\alpha^+} \\ &\quad - n(n-1)(\gamma^{\alpha_1})^{n-2} V_{-\alpha^-}^{\alpha_1} \left( (\gamma^{\alpha_1})^2 E_{\alpha_1} + \gamma^{\alpha_1} H_1 - F_{\alpha_1} \right) \\ &\quad + n(n-1)(\gamma^{\alpha_1})^{n-2} \sum_{\nu, \sigma > 0} \left( (\gamma^{\alpha_1})^2 V_{\alpha_1}^\nu + \gamma^{\alpha_1} V_1^\nu - V_{-\alpha_1}^\nu \right) \partial_\nu \partial_\sigma V_{-\alpha^-}^{\alpha_1} \partial \gamma^\sigma \end{aligned} \quad (19)$$

Their anti-commutators are

$$\{G_{\alpha^-; n+1/2}, G_{\beta^-; m+1/2}\} = 0 \quad (20)$$

and

$$\{G_{\alpha^-; n+1/2}, \overline{G}_{-\beta^-; m-1/2}\} = \delta_{\alpha^-, \beta^-} L_{n+m} + (n-m+1) K_{\alpha^-; -\beta^-; n+m} + \frac{1}{6} c n(n+1) \delta_{n+m, 0} \delta_{\alpha^-, \beta^-} \quad (21)$$

where the primary spin 1 (with respect to (14)) current  $K$  is defined by

$$\begin{aligned} K_{\alpha^-; -\beta^-; n} &= \oint \frac{dz}{2\pi i} \mathcal{K}_{\alpha^-; -\beta^-; n}(z) \\ \mathcal{K}_{\alpha^-; -\beta^-; n} &= n(\gamma^{\alpha_1})^{n-1} V_{-\beta^-}^{\alpha_1} (\gamma^{\alpha_1} E_{\alpha^+} - E_{\alpha^-}) - (\gamma^{\alpha_1})^n f_{\alpha^-, -\beta^-}{}^c J_c \\ &\quad + \frac{1}{2} \delta_{\alpha^-, \beta^-} (\gamma^{\alpha_1})^{n-1} \left( n \left( (\gamma^{\alpha_1})^2 E_{\alpha_1} + \gamma^{\alpha_1} H_1 - F_{\alpha_1} \right) - \gamma^{\alpha_1} H_1 \right) \\ &\quad + n(\gamma^{\alpha_1})^{n-1} \sum_{\nu, \sigma > 0} (\gamma^{\alpha_1} V_{\alpha^+}^\nu - V_{\alpha^-}^\nu) \partial_\nu \partial_\sigma V_{-\beta^-}^{\alpha_1} \partial \gamma^\sigma \end{aligned} \quad (22)$$

It was also shown in Ref. [4], that the remaining anti-commutators are generally non-vanishing:

$$\{\overline{G}_{-\alpha^-;n-1/2}, \overline{G}_{-\beta^-;m-1/2}\} \neq 0, \quad \text{for } \alpha^- \neq \beta^-, n \neq m, n+m \neq 1 \quad (23)$$

This asymmetric property of the SCAs induced by  $SL(2|N/2)$  will be illustrated in the following where we consider the case  $N = 4$ .

### 3 $N = 4$ Superconformal Algebras

Here we shall specialize the general considerations on  $SL(2|N/2)$  in Section 2 to the case  $N = 4$ . The resulting SCA is of a new and asymmetric form. Its classical and centerless analogue has recently been obtained in Ref. [4]. Below we shall provide the full SCA for arbitrary central charge. We shall furthermore show that substituting the original  $SL(2|2)$  by the coset  $SL(2|2)/U(1)$  reduces the asymmetric  $N = 4$  SCA to the well known small  $N = 4$  SCA. An invariance of the asymmetric  $N = 4$  SCA is also presented.

#### 3.1 Asymmetric $N = 4$ SCA from $SL(2|2)$

In the distinguished representation discussed in Appendix A, the Cartan matrix and the Cartan-Killing form of the Lie superalgebra  $SL(2|2)$  are

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \kappa_{i,j} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix} \quad (24)$$

The dual Coxeter number is  $h^\vee = 0$  while the number of supercurrents is  $2|\Delta_+^1| = 4$ . Accordingly, there are a priori 4 generators,  $K$ , of the internal Kac-Moody algebra. Using these facts, with the explicit realizations of the Virasoro generators (14), the supercurrents (18) and (19), and the affine currents (22) (given in Appendix C), one may work out the entire SCA. We find that closure is ensured by the following set of generators

Virasoro generator	$L$	$h = 2$
supercurrents	$G_{\alpha_2}, G_{\alpha_2+\alpha_3}, \overline{G}_{-\alpha_2}, \overline{G}_{-(\alpha_2+\alpha_3)}$	$h = 3/2$
affine $SL(2)$	$\tilde{E} = K_{\alpha_2+\alpha_3;-\alpha_2},$ $\tilde{H} = K_{\alpha_2+\alpha_3;-(\alpha_2+\alpha_3)} - K_{\alpha_2;-\alpha_2},$ $\tilde{F} = K_{\alpha_2;-(\alpha_2+\alpha_3)}$	$h = 1$
affine $U(1)$	$U = K_{\alpha_2+\alpha_3;-(\alpha_2+\alpha_3)} + K_{\alpha_2;-\alpha_2}$	$h = 1$
fermions	$\phi_{-\alpha_2}, \phi_{-(\alpha_2+\alpha_3)}$	$h = 1/2$
scalar	$S$	$h = 0$

(25)

and that the non-trivial (anti-)commutators are

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

$$\begin{aligned}
[L_n, A_m] &= ((h(A) - 1)n - m)A_{n+m} \\
\{G_{\alpha^-; n+1/2}, G_{\beta^-; m+1/2}\} &= \{\bar{G}_{-\alpha^-; n-1/2}, \bar{G}_{-\alpha^-; m-1/2}\} = 0 \\
\{G_{\alpha^-; n+1/2}, \bar{G}_{-\beta^-; m-1/2}\} &= \delta_{\alpha^-, \beta^-} L_{n+m} + (n - m + 1)K_{\alpha^-; -\beta^-; n+m} \\
&\quad + \frac{1}{6}cn(n+1)\delta_{n+m,0}\delta_{\alpha^-, \beta^-} \\
\{\bar{G}_{-\alpha_2; n-1/2}, \bar{G}_{-(\alpha_2+\alpha_3); m-1/2}\} &= (n - m)(n + m - 1)S_{n+m-1} \\
[\tilde{H}_n, \tilde{E}_m] &= 2\tilde{E}_{n+m}, \quad [\tilde{H}_n, \tilde{F}_m] = -2\tilde{F}_{n+m} \\
[\tilde{E}_n, \tilde{F}_m] &= \tilde{H}_{n+m} + \frac{1}{6}cn\delta_{n+m,0}, \quad [\tilde{H}_n, \tilde{H}_m] = \frac{1}{3}cn\delta_{n+m,0} \\
[\tilde{E}_n, G_{\alpha_2; m+1/2}] &= G_{\alpha_2+\alpha_3; n+m+1/2}, \quad [\tilde{F}_n, G_{\alpha_2+\alpha_3; m+1/2}] = G_{\alpha_2; n+m+1/2} \\
[\tilde{H}_n, G_{\alpha_2; m+1/2}] &= -G_{\alpha_2; n+m+1/2} \\
[\tilde{H}_n, G_{\alpha_2+\alpha_3; m+1/2}] &= G_{\alpha_2+\alpha_3; n+m+1/2} \\
[\tilde{E}_n, \bar{G}_{-(\alpha_2+\alpha_3); m-1/2}] &= -\bar{G}_{-\alpha_2; n+m-1/2} - n\phi_{-\alpha_2; n+m-1/2} \\
[\tilde{H}_n, \bar{G}_{-\alpha_2; m-1/2}] &= \bar{G}_{-\alpha_2; n+m-1/2} + n\phi_{-\alpha_2; n+m-1/2} \\
[\tilde{H}_n, \bar{G}_{-(\alpha_2+\alpha_3); m-1/2}] &= -\bar{G}_{-(\alpha_2+\alpha_3); n+m-1/2} - n\phi_{-(\alpha_2+\alpha_3); n+m-1/2} \\
[\tilde{F}_n, \bar{G}_{-\alpha_2; m-1/2}] &= -\bar{G}_{-(\alpha_2+\alpha_3); n+m-1/2} - n\phi_{-(\alpha_2+\alpha_3); n+m-1/2} \\
[U_n, \bar{G}_{-\alpha^-; m-1/2}] &= n\phi_{-\alpha^-; n+m-1/2} \\
[S_n, G_{\alpha_2; m+1/2}] &= \phi_{-(\alpha_2+\alpha_3); n+m+1/2} \\
[S_n, G_{\alpha_2+\alpha_3; m+1/2}] &= -\phi_{-\alpha_2; n+m+1/2} \\
\{G_{\alpha_2; n+1/2}, \phi_{-\alpha_2; m-1/2}\} &= U_{n+m}, \quad \{G_{\alpha_2+\alpha_3; n+1/2}, \phi_{-(\alpha_2+\alpha_3); m-1/2}\} = U_{n+m} \\
\{\bar{G}_{-\alpha_2; n-1/2}, \phi_{-(\alpha_2+\alpha_3); m-1/2}\} &= (n + m - 1)S_{n+m-1} \\
\{\bar{G}_{-(\alpha_2+\alpha_3); n-1/2}, \phi_{-\alpha_2; m-1/2}\} &= -(n + m - 1)S_{n+m-1} \\
[\tilde{E}_n, \phi_{-(\alpha_2+\alpha_3); m-1/2}] &= -\phi_{-\alpha_2; n+m-1/2} \\
[\tilde{F}_n, \phi_{-\alpha_2; m-1/2}] &= -\phi_{-(\alpha_2+\alpha_3); n+m-1/2} \\
[\tilde{H}_n, \phi_{-\alpha_2; m-1/2}] &= \phi_{-\alpha_2; n+m-1/2} \\
[\tilde{H}_n, \phi_{-(\alpha_2+\alpha_3); m-1/2}] &= -\phi_{-(\alpha_2+\alpha_3); n+m-1/2} \\
[U_n, U_m] &= [S_n, S_m] = 0
\end{aligned} \tag{26}$$

$A_m$  denotes any of the 11 generators listed in (25) different from the Virasoro generator. In the derivation we have used that integrating a total derivative gives zero. In particular, we find

$$\begin{aligned}
&\{\bar{G}_{-\alpha_2; n-1/2}, \bar{G}_{-(\alpha_2+\alpha_3); m-1/2}\} \\
&= (n - m)(n + m - 1) \left( S_{n+m-1} + \oint \frac{dz}{2\pi i} \frac{\partial}{\partial z} \left[ \gamma^{n+m-2}(z) V_{-\alpha_2}^{\alpha_1}(\gamma(z)) V_{-(\alpha_2+\alpha_3)}^{\alpha_1}(\gamma(z)) \right] \right)
\end{aligned}$$

$$= (n-m)(n+m-1)S_{n+m-1} \quad (27)$$

The polynomials  $V_{-\alpha_2}^{\alpha_1}$  and  $V_{-(\alpha_2+\alpha_3)}^{\alpha_1}$  are given (55) in Appendix C.

It is observed that this  $N=4$  SCA (26) is a new and asymmetric one not contained in the standard classification of  $N=4$  SCAs [8, 9, 10, 11, 12]. Besides being asymmetric in the way the  $G$  and  $\overline{G}$  supercurrents are treated, it involves the unfamiliar number *two* of spin 1/2 fermions. We recall that the small  $N=4$  SCA does not contain any spin 1/2 fermions, whereas the big  $N=4$  SCAs are characterized by containing 4 such generators.

### 3.2 Small $N=4$ SCA from Coset $SL(2|2)/U(1)$

Among the Lie superalgebras  $sl(2|M)$ ,  $M \geq 1$ ,  $sl(2|2)$  is the only one having a non-trivial center. This may be seen easily by considering the associated Cartan matrices; they are all invertible except when  $M=2$ . By simple inspection of the Cartan matrix for  $sl(2|2)$  (24) it follows that the Lie superalgebra element

$$H_{u(1)} = H_1 + 2H_2 - H_3 \quad (28)$$

generates the center  $u(1)$  of  $sl(2|2)$ . The coset algebra  $sl(2|2)/u(1)$  may therefore be realized straightforwardly in terms of the generators of the original  $sl(2|2)$ . All that one has to do is to invoke the vanishing of the generator of the center (28), and otherwise make no changes. It should be noted that the root space of the coset algebra is identified with the root space of  $sl(2|2)$ . In particular, the sets of simple roots are identical, despite the fact that the rank of the coset algebra is one smaller than the rank of  $sl(2|2)$ , the latter being  $r=3$ .

The associated affine  $SL(2|2)/U(1)$  current superalgebra is equally simple to realize. By construction, its Virasoro generator  $T_{SL(2|2)/U(1)} = T_{SL(2|2)} - T_{U(1)}$  has vanishing OPE with the current

$$\begin{aligned} H_{U(1)}(z) &= H_1(z) + 2H_2(z) - H_3(z) \\ &= \sqrt{k}(\partial\varphi_1(z) + 2\partial\varphi_2(z) - \partial\varphi_3(z)) \end{aligned} \quad (29)$$

while  $T_{U(1)}$  has vanishing OPEs with all other affine currents. Thus, it makes sense from the conformal field theory point of view to put the central current (29) equal to zero without modifying the Virasoro generator of the original  $SL(2|2)$  current superalgebra, but by imposing the (coset) condition

$$\partial\varphi_1 + 2\partial\varphi_2 - \partial\varphi_3 \equiv 0 \quad (30)$$

In this way the coset current superalgebra has the same free field realization as the original current superalgebra, though subject to the coset condition (30). Thus, at the level of the free field realization the coset condition (30) reflects modding out the  $U(1)$  center of  $SL(2|2)$ .

From the explicit realization of the generators of the asymmetric  $N=4$  SCA induced by  $SL(2|2)$  (see Appendix C) it follows that imposing the coset condition (30) has fundamental consequences for the resulting  $N=4$  SCA. Even the number of generators is



reduced as the affine  $U(1)$  subalgebra generated by  $U$  (52), the 2 spin 1/2 fermions  $\phi$  (53), and the bosonic scalar  $S$  (54) all vanish identically

$$U_n \equiv \phi_{-\alpha_2;n-1/2} \equiv \phi_{-(\alpha_2+\alpha_3);n-1/2} \equiv S_n \equiv 0 \quad (31)$$

We conclude that the  $N = 4$  SCA induced by the affine  $SL(2|2)/U(1)$  current superalgebra is generated by

$$\begin{array}{llll} \text{Virasoro generator} & L & h = 2 \\ \text{supercurrents} & G_{\alpha_2}, G_{\alpha_2+\alpha_3}, \bar{G}_{-\alpha_2}, \bar{G}_{-(\alpha_2+\alpha_3)} & h = 3/2 \\ \text{affine } SL(2) & \tilde{E} = K_{\alpha_2+\alpha_3;-\alpha_2}, \\ & \tilde{H} = K_{\alpha_2+\alpha_3;-(\alpha_2+\alpha_3)} - K_{\alpha_2;-\alpha_2}, \\ & \tilde{F} = K_{\alpha_2;-(\alpha_2+\alpha_3)} & h = 1 \end{array} \quad (32)$$

where  $U = K_{\alpha_2+\alpha_3;-(\alpha_2+\alpha_3)} + K_{\alpha_2;-\alpha_2} = 0$ . The non-trivial (anti-)commutators are

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \\ [L_n, A_m] &= ((h(A)-1)n-m)A_{n+m} \\ \{G_{\alpha^-;n+1/2}, G_{\beta^-;m+1/2}\} &= \{\bar{G}_{-\alpha^-;n-1/2}, \bar{G}_{-\beta^-;m-1/2}\} = 0 \\ \{G_{\alpha^-;n+1/2}, \bar{G}_{-\beta^-;m-1/2}\} &= \delta_{\alpha^-, \beta^-} L_{n+m} + (n-m+1)K_{\alpha^-;-\beta^-;n+m} \\ &\quad + \frac{1}{6}cn(n+1)\delta_{n+m,0}\delta_{\alpha^-, \beta^-} \\ [\tilde{E}_n, G_{\alpha_2;m+1/2}] &= G_{\alpha_2+\alpha_3;n+m+1/2}, \quad [\tilde{F}_n, G_{\alpha_2+\alpha_3;m+1/2}] = G_{\alpha_2;n+m+1/2} \\ [\tilde{H}_n, G_{\alpha_2;m+1/2}] &= -G_{\alpha_2;n+m+1/2}, \quad [\tilde{H}_n, G_{\alpha_2+\alpha_3;m+1/2}] = G_{\alpha_2+\alpha_3;n+m+1/2} \\ [\tilde{E}_n, \bar{G}_{-(\alpha_2+\alpha_3);m-1/2}] &= -\bar{G}_{-\alpha_2;n+m-1/2} \\ [\tilde{F}_n, \bar{G}_{-\alpha_2;m-1/2}] &= -\bar{G}_{-(\alpha_2+\alpha_3);n+m-1/2} \\ [\tilde{H}_n, \bar{G}_{-\alpha_2;m-1/2}] &= \bar{G}_{-\alpha_2;n+m-1/2} \\ [\tilde{H}_n, \bar{G}_{-(\alpha_2+\alpha_3);m-1/2}] &= -\bar{G}_{-(\alpha_2+\alpha_3);n+m-1/2} \\ [\tilde{H}_n, \tilde{E}_m] &= 2\tilde{E}_{n+m}, \quad [\tilde{H}_n, \tilde{F}_m] = -2\tilde{F}_{n+m} \\ [\tilde{E}_n, \tilde{F}_m] &= \tilde{H}_{n+m} + \frac{1}{6}cn\delta_{n+m,0}, \quad [\tilde{H}_n, \tilde{H}_m] = \frac{1}{3}cn\delta_{n+m,0} \end{aligned} \quad (33)$$

$A_m$  denotes any of the 7 generators listed in (32) different from the Virasoro generator. The SCA (33) is recognized as the well known small  $N = 4$  SCA thus proving our assertion.

### 3.3 An Invariance and a Reduction

It turns out that the new and asymmetric  $N = 4$  SCA possesses an invariance which may be revealed by considering the consequences of modifying the Virasoro generators according to

$$L_n^\lambda = L_n + \lambda(n+1)U_n \quad (34)$$

with  $\lambda$  arbitrary. The Virasoro algebra is easily seen to be generated with unchanged central charge

$$c^\lambda = c \quad (35)$$

We find that the asymmetric  $N = 4$  SCA (26) is invariant under the following one-parameter twist of its generators

$$\begin{aligned} L_n^\lambda &= L_n + \lambda(n+1)U_n \\ G_{\alpha^-; n+1/2}^\lambda &= G_{\alpha^-; n+1/2} \\ \overline{G}_{-\alpha^-; n-1/2}^\lambda &= \overline{G}_{-\alpha^-; n-1/2} + 2\lambda n \phi_{-\alpha^-; n-1/2} \\ K_{\alpha^-; -\beta^-; n}^\lambda &= K_{\alpha^-; -\beta^-; n} - \lambda \delta_{\alpha^-, \beta^-} U_n \\ \phi_{-\alpha^-; n-1/2}^\lambda &= (1-2\lambda)\phi_{-\alpha^-; n-1/2} \\ S_n^\lambda &= (1-2\lambda)S_n \end{aligned} \quad (36)$$

In particular, the 11 generators besides  $L^\lambda$  are all primary of unchanged weights

$$[L_n^\lambda, A_m^\lambda] = ((h(A) - 1)n - m)A_{n+m}^\lambda \quad (37)$$

It should be mentioned that one may obtain the twisted supercurrents by following the exact same procedure which leads to the construction of the untwisted supercurrents, see Ref. [4]. In particular, we have the relations

$$\begin{aligned} \left[ L_n^\lambda, \oint \frac{dz}{2\pi i} E_{\alpha^-} \right] &= \frac{1}{2}(n-1)G_{\alpha^-; n+1/2}^\lambda \\ \left[ L_n^\lambda, \oint \frac{dz}{2\pi i} F_{\alpha^-} \right] &= -\frac{1}{2}(n+1)\overline{G}_{-\alpha^-; n-1/2}^\lambda \end{aligned} \quad (38)$$

The remaining twisted generators may of course be found by working out the relevant (anti-)commutators of twisted generators already obtained, and requiring the algebra to be invariant. For the  $K$  generators, let us write out the result of the twisting (36)

$$\begin{aligned} \tilde{E}_n^\lambda &= \tilde{E}_n, & \tilde{H}_n^\lambda &= \tilde{H}_n, & \tilde{F}_n^\lambda &= \tilde{F}_n \\ U_n^\lambda &= (1-2\lambda)U_n \end{aligned} \quad (39)$$

We observe that for the unique value

$$\lambda = 1/2 \quad (40)$$

twisting (36) is not an invariance but rather a reduction. The 4 generators  $U^{\lambda=1/2}$ ,  $\phi_{-\alpha^-}^{\lambda=1/2}$  and  $S^{\lambda=1/2}$  all vanish identically, and the resulting SCA is instead the small  $N = 4$  SCA.

So far we have not addressed the question of BRST invariance of our construction of the  $N = 4$  SCAs. As pointed out in Ref. [6], BRST invariance of the construction of the space-time conformal algebra based on a world sheet  $SL(2)$  current algebra, is equivalent to requiring the Virasoro generators to be primary fields of weight one with respect to the world sheet energy-momentum tensor, ensuring that the integrated fields commute with the world sheet Virasoro algebra. This carries over to the superconformal case where all

the generators are required to be integrated spin one primary fields with respect to the world sheet Virasoro algebra. In Ref. [4] we have shown that all the generators of the  $SL(2|2)$  induced SCA are indeed BRST invariant. Since the twisted generators (36) are linear combinations of those fields, they are themselves BRST invariant. The generators of the small  $N = 4$  SCA induced by the coset  $SL(2|2)/U(1)$  in Section 3.2, are also BRST invariant. This follows immediately, as the construction is based on the same free field realization as the original  $SL(2|2)$ , the only (and in this respect trivial) difference being the coset condition (30).

## 4 Conclusion

We believe that the general construction of SCAs outlined in Ref. [4] (and indicated in Ref. [5]) is interesting from a mathematical as well as from a string theoretical point of view. A mathematical or conformal field theoretical interest lies in the fact that besides providing new realizations of well known SCAs, the construction also produces whole new classes of SCAs. An additional virtue is that the SCAs are realized explicitly. The asymmetric  $N = 4$  SCA discussed in the present paper is an example of such a new SCA. There are also convincing indications that new bosonic extensions of the Virasoro algebra may be obtained using a modification of the construction. This will be the subject of a forthcoming publication.

The construction is interesting from the string theoretical point of view, as it produces the unique boundary SCA associated to a string theory on  $AdS_3$  with a certain affine Lie supergroup symmetry. As already pointed out, this pertains to Lie supergroups with  $SL(2) \otimes G'$  decomposable bosonic subgroups. Due to the recently discovered  $AdS/CFT$  correspondence, the question of determining which superconformal field theory is associated to which string theory has become increasingly relevant. We hope that our work will add to the understanding of this.

## Acknowledgment

The author thanks Spenta Wadia and Mark Walton for comments. The author is also grateful to The Niels Bohr Institute, where part of this work was done, for its kind hospitality. He is supported in part by NSERC of Canada.

## A Lie Superalgebra $sl(2|M)$

The root space of the Lie superalgebra  $sl(2|M)$  in the distinguished representation may be realized in terms of an orthonormal two-dimensional basis  $\{\epsilon_1, \epsilon_2\}$  and an orthonormal  $M$ -dimensional basis  $\{\delta_u\}_{u=1,\dots,M}$  with metrics

$$\epsilon_i \cdot \epsilon_{i'} = \delta_{i,i'} \ , \quad \delta_u \cdot \delta_{u'} = -\delta_{u,u'} \ , \quad \epsilon_i \cdot \delta_u = 0 \quad (41)$$

The  $\frac{1}{2}(M+1)(M+2)$  positive roots are then represented as

$$\Delta_+^0 = \{\epsilon_1 - \epsilon_2\} \cup \{\delta_u - \delta_v \mid u < v\}$$

$$\begin{aligned}
\Delta_+^{1+} &= \{\epsilon_1 - \delta_u \mid u = 1, \dots, M\} \\
\Delta_+^{1-} &= \{\epsilon_2 - \delta_u \mid u = 1, \dots, M\}
\end{aligned} \tag{42}$$

where the  $M + 1$  simple roots  $\alpha_i$  are

$$\begin{aligned}
\alpha_1 &= \epsilon_1 - \epsilon_2 \\
\alpha_2 &= \epsilon_2 - \delta_1 \\
\alpha_{u+2} &= \delta_u - \delta_{u+1}
\end{aligned} \tag{43}$$

The associated ladder operators  $E_\alpha$  and  $F_\alpha$ , and the Cartan generators  $H_i$  admit a standard oscillator realization (see e.g. [15])

$$\begin{aligned}
E_{\epsilon_1 - \epsilon_2} &= a_1^\dagger a_2, & E_{\epsilon_1 - \delta_u} &= a_u^\dagger b_u, & E_{\delta_u - \delta_v} &= b_u^\dagger b_v \\
F_{\epsilon_1 - \epsilon_2} &= a_2^\dagger a_1, & F_{\epsilon_1 - \delta_u} &= b_u^\dagger a_u, & F_{\delta_u - \delta_v} &= b_v^\dagger b_u \\
H_1 &= a_1^\dagger a_1 - a_2^\dagger a_2, & H_2 &= a_2^\dagger a_2 + b_1^\dagger b_1, & H_{u+2} &= b_u^\dagger b_u - b_{u+1}^\dagger b_{u+1}
\end{aligned} \tag{44}$$

where  $a_\iota^{(\dagger)}$  and  $b_u^{(\dagger)}$  are fermionic and bosonic oscillators, respectively, satisfying

$$\{a_\iota, a_{\iota'}^\dagger\} = \delta_{\iota, \iota'}, \quad [b_u, b_v^\dagger] = \delta_{u, v}, \quad [b_u^{(\dagger)}, a_{\iota'}^{(\dagger)}] = 0 \tag{45}$$

## B Free Field Realization of Affine $SL(2|2)$ Current Superalgebra

In this appendix we shall provide the explicit free field realization of the affine  $SL(2|2)$  current superalgebra that is discussed in Section 2. Recall that the only bosonic ghost fields are  $\gamma^{\alpha_1}, \gamma^{\alpha_3}, \beta_{\alpha_1}, \beta_{\alpha_3}$ . Let us introduce the abbreviations  $\gamma^1, \gamma^{23}, \beta_{123}, \dots$  for the ghost fields  $\gamma^{\alpha_1}, \gamma^{\alpha_2 + \alpha_3}, \beta_{\alpha_1 + \alpha_2 + \alpha_3}, \dots$ . The free field realization is

$$\begin{aligned}
E_{\alpha_1} &= \beta_1 - \frac{1}{2}\gamma^2\beta_{12} + \frac{1}{2}\left(\frac{1}{6}\gamma^2\gamma^3 - \gamma^{23}\right)\beta_{123} \\
E_{\alpha_2} &= \beta_2 + \frac{1}{2}\gamma^1\beta_{12} - \frac{1}{2}\gamma^3\beta_{23} - \frac{1}{6}\gamma^1\gamma^3\beta_{123} \\
E_{\alpha_3} &= \beta_3 + \frac{1}{2}\gamma^2\beta_{23} + \frac{1}{2}\left(\frac{1}{6}\gamma^1\gamma^2 + \gamma^{12}\right)\beta_{123} \\
E_{\alpha_1 + \alpha_2} &= \beta_{12} - \frac{1}{2}\gamma^3\beta_{123} \\
E_{\alpha_2 + \alpha_3} &= \beta_{23} + \frac{1}{2}\gamma^1\beta_{123} \\
E_{\alpha_1 + \alpha_2 + \alpha_3} &= \beta_{123} \\
H_1 &= -2\gamma^1\beta_1 + \gamma^2\beta_2 - \gamma^{12}\beta_{12} + \gamma^{23}\beta_{23} - \gamma^{123}\beta_{123} + \sqrt{k}\partial\varphi_1 \\
H_2 &= \gamma^1\beta_1 - \gamma^3\beta_3 + \gamma^{12}\beta_{12} - \gamma^{23}\beta_{23} + \sqrt{k}\partial\varphi_2 \\
H_3 &= \gamma^2\beta_2 - 2\gamma^3\beta_3 + \gamma^{12}\beta_{12} - \gamma^{23}\beta_{23} - \gamma^{123}\beta_{123} + \sqrt{k}\partial\varphi_3 \\
F_{\alpha_1} &= -(\gamma^1)^2\beta_1 + \left(\frac{1}{2}\gamma^1\gamma^2 - \gamma^{12}\right)\beta_2 - \frac{1}{2}\gamma^1\left(\frac{1}{2}\gamma^1\gamma^2 + \gamma^{12}\right)\beta_{12}
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{1}{12} \gamma^1 \gamma^2 \gamma^3 + \frac{1}{2} \gamma^1 \gamma^{23} - \gamma^{123} \right) \beta_{23} - \frac{1}{2} \gamma^1 \left( \frac{1}{2} \gamma^1 \gamma^{23} + \frac{1}{6} \gamma^{12} \gamma^3 + \gamma^{123} \right) \beta_{123} \\
& + \gamma^1 \sqrt{k} \partial \varphi_1 + (k-1) \partial \gamma^1 \\
F_{\alpha_2} & = \left( \frac{1}{2} \gamma^1 \gamma^2 + \gamma^{12} \right) \beta_1 - \left( \frac{1}{2} \gamma^2 \gamma^3 - \gamma^{23} \right) \beta_3 + \frac{1}{2} \gamma^2 \gamma^{12} \beta_{12} \\
& - \frac{1}{2} \gamma^2 \gamma^{23} \beta_{23} + \frac{1}{6} \gamma^2 \left( \gamma^1 \gamma^{23} + \gamma^{12} \gamma^3 \right) \beta_{123} + \gamma^2 \sqrt{k} \partial \varphi_2 + k \partial \gamma^2 \\
F_{\alpha_3} & = \left( \frac{1}{2} \gamma^2 \gamma^3 + \gamma^{23} \right) \beta_2 - (\gamma^3)^2 \beta_3 - \left( \frac{1}{12} \gamma^1 \gamma^2 \gamma^3 - \frac{1}{2} \gamma^{12} \gamma^3 - \gamma^{123} \right) \beta_{12} \\
& + \frac{1}{2} \gamma^3 \left( \frac{1}{2} \gamma^2 \gamma^3 - \gamma^{23} \right) \beta_{23} + \frac{1}{2} \gamma^3 \left( \frac{1}{6} \gamma^1 \gamma^{23} + \frac{1}{2} \gamma^{12} \gamma^3 - \gamma^{123} \right) \beta_{123} \\
& + \gamma^3 \sqrt{k} \partial \varphi_3 - (k+1) \partial \gamma^3 \\
F_{\alpha_1+\alpha_2} & = -\gamma^1 \left( \frac{1}{2} \gamma^1 \gamma^2 + \gamma^{12} \right) \beta_1 - \gamma^2 \gamma^{12} \beta_2 \\
& + \left( \frac{1}{6} \gamma^1 \gamma^2 \gamma^3 - \frac{1}{2} \gamma^1 \gamma^{23} - \frac{1}{2} \gamma^{12} \gamma^3 + \gamma^{123} \right) \beta_3 \\
& + \frac{1}{2} \gamma^2 \left( \frac{1}{2} \gamma^1 \gamma^{23} - \frac{1}{2} \gamma^{12} \gamma^3 - \gamma^{123} \right) \beta_{23} \\
& - \frac{1}{2} \left( \frac{5}{12} (\gamma^1)^2 \gamma^2 \gamma^{23} + \frac{1}{4} \gamma^1 \gamma^2 \gamma^{12} \gamma^3 + \frac{1}{6} \gamma^1 \gamma^2 \gamma^{123} + \frac{1}{2} \gamma^1 \gamma^{12} \gamma^{23} + \gamma^{12} \gamma^{123} \right) \beta_{123} \\
& + \left( \frac{1}{2} \gamma^1 \gamma^2 + \gamma^{12} \right) \sqrt{k} \partial \varphi_1 - \left( \frac{1}{2} \gamma^1 \gamma^2 - \gamma^{12} \right) \sqrt{k} \partial \varphi_2 \\
& + \frac{6k-11}{12} \gamma^2 \partial \gamma^1 - \frac{3k+1}{6} \gamma^1 \partial \gamma^2 + (k-1/2) \partial \gamma^{12} \\
F_{\alpha_2+\alpha_3} & = \left( \frac{1}{6} \gamma^1 \gamma^2 \gamma^3 + \frac{1}{2} \gamma^1 \gamma^{23} + \frac{1}{2} \gamma^{12} \gamma^3 + \gamma^{123} \right) \beta_1 + \gamma^2 \gamma^{23} \beta_2 \\
& - \gamma^3 \left( \frac{1}{2} \gamma^2 \gamma^3 - \gamma^{23} \right) \beta_3 - \frac{1}{2} \gamma^2 \left( \frac{1}{2} \gamma^1 \gamma^{23} - \frac{1}{2} \gamma^{12} \gamma^3 - \gamma^{123} \right) \beta_{12} \\
& + \frac{1}{2} \left( \frac{1}{4} \gamma^1 \gamma^2 \gamma^{23} \gamma^3 + \frac{5}{12} \gamma^2 \gamma^{12} (\gamma^3)^2 - \frac{1}{6} \gamma^2 \gamma^{123} \gamma^3 + \frac{1}{2} \gamma^{12} \gamma^{23} \gamma^3 + \gamma^{23} \gamma^{123} \right) \beta_{123} \\
& + \left( \frac{1}{2} \gamma^2 \gamma^3 + \gamma^{23} \right) \sqrt{k} \partial \varphi_2 + \left( \frac{1}{2} \gamma^2 \gamma^3 - \gamma^{23} \right) \sqrt{k} \partial \varphi_3 \\
& + \frac{3k-1}{6} \gamma^3 \partial \gamma^2 - \frac{6k+11}{12} \gamma^2 \partial \gamma^3 + (k+1/2) \partial \gamma^{23} \\
F_{\alpha_1+\alpha_2+\alpha_3} & = -\gamma^1 \left( \frac{1}{6} \gamma^1 \gamma^2 \gamma^3 + \frac{1}{2} \gamma^1 \gamma^{23} + \frac{1}{2} \gamma^{12} \gamma^3 + \gamma^{123} \right) \beta_1 \\
& - \left( \frac{1}{2} \gamma^1 \gamma^2 \gamma^{23} + \frac{1}{2} \gamma^2 \gamma^{12} \gamma^3 - \gamma^{12} \gamma^{23} \right) \beta_2 \\
& + \gamma^3 \left( \frac{1}{6} \gamma^1 \gamma^2 \gamma^3 - \frac{1}{2} \gamma^1 \gamma^{23} - \frac{1}{2} \gamma^{12} \gamma^3 + \gamma^{123} \right) \beta_3 \\
& + \left( \frac{1}{4} (\gamma^1)^2 \gamma^2 \gamma^{23} + \frac{1}{12} \gamma^1 \gamma^2 \gamma^{12} \gamma^3 + \gamma^{12} \gamma^{123} \right) \beta_{12} \\
& - \left( \frac{1}{4} \gamma^2 \gamma^{12} (\gamma^3)^2 + \frac{1}{12} \gamma^1 \gamma^2 \gamma^{23} \gamma^3 + \gamma^{23} \gamma^{123} \right) \beta_{23}
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{6}\gamma^1\gamma^3 \left( \frac{1}{2}\gamma^1\gamma^2\gamma^{23} + \frac{1}{2}\gamma^2\gamma^{12}\gamma^3 + \gamma^{12}\gamma^{23} \right) \beta_{123} \\
& + \left( \frac{1}{6}\gamma^1\gamma^2\gamma^3 + \frac{1}{2}\gamma^1\gamma^{23} + \frac{1}{2}\gamma^{12}\gamma^3 + \gamma^{123} \right) \sqrt{k}\partial\varphi_1 \\
& - \left( \frac{1}{3}\gamma^1\gamma^2\gamma^3 + \frac{1}{2}\gamma^1\gamma^{23} - \frac{1}{2}\gamma^{12}\gamma^3 - \gamma^{123} \right) \sqrt{k}\partial\varphi_2 \\
& - \left( \frac{1}{6}\gamma^1\gamma^2\gamma^3 - \frac{1}{2}\gamma^1\gamma^{23} - \frac{1}{2}\gamma^{12}\gamma^3 + \gamma^{123} \right) \sqrt{k}\partial\varphi_3 \\
& + \frac{2k-5}{12}\gamma^2\gamma^3\partial\gamma^1 + \frac{k-2}{2}\gamma^{23}\partial\gamma^1 - \frac{k}{3}\gamma^1\gamma^3\partial\gamma^2 + \frac{2k+5}{12}\gamma^1\gamma^2\partial\gamma^3 \\
& - \frac{k+2}{2}\gamma^{12}\partial\gamma^3 + \frac{k-1}{2}\gamma^3\partial\gamma^{12} - \frac{k+1}{2}\gamma^1\partial\gamma^{23} + k\partial\gamma^{123}
\end{aligned} \tag{46}$$

## C Generators of Asymmetric $N = 4$ SCA

For completeness and reference, below are listed explicit free field realizations of the integrands  $\mathcal{A}_n$  of the generators  $A_n$  (25) of the asymmetric  $N = 4$  SCA

$$A = \oint \frac{dz}{2\pi i} \mathcal{A}_n(z) \tag{47}$$

The realizations are obtained by inserting the results of Appendix B in the general expressions for the generators provided in Section 3. The realizations of  $\phi_{-\alpha_2}$ ,  $\phi_{-(\alpha_2+\alpha_3)}$  and  $S$  may be obtained by working out explicitly the relevant (anti-)commutators of the asymmetric  $N = 4$  SCA (26).

The Virasoro generator is realized as

$$\begin{aligned}
\mathcal{L}_n = & -(\gamma^1)^{n+1}\beta_1 - \frac{n+1}{2}(\gamma^1)^{n-1} \left( \frac{n-2}{2}\gamma^1\gamma^2 + n\gamma^{12} \right) \beta_2 \\
& + (\gamma^1)^n \left( \frac{n(n-3)}{8}\gamma^1\gamma^2 + \frac{n^2-n-2}{4}\gamma^{12} \right) \beta_{12} \\
& + \frac{n+1}{2}(\gamma^1)^{n-1} \left( \frac{n}{12}\gamma^1\gamma^2\gamma^3 - \frac{n-2}{2}\gamma^1\gamma^{23} - n\gamma^{123} \right) \beta_{23} \\
& - (\gamma^1)^n \left( \frac{n(n-1)}{24}\gamma^1\gamma^2\gamma^3 - \frac{n(n-3)}{8}\gamma^1\gamma^{23} + \frac{n(n+1)}{24}\gamma^{12}\gamma^3 \right. \\
& \quad \left. - \frac{(n+1)(n-2)}{4}\gamma^{123} \right) \beta_{123} + \frac{1}{2}(n+1)(\gamma^1)^n \sqrt{k}\partial\varphi_1
\end{aligned} \tag{48}$$

Here and throughout Appendix C we have taken advantage of the fact that terms proportional to  $n(\gamma^1)^{n-1}\partial\gamma^1$  vanish upon integration. They are not included in the expressions for the integrands. The supercurrents are realized as

$$\begin{aligned}
\mathcal{G}_{\alpha_2; n+1/2} &= (\gamma^1)^n \left( (n+1)\beta_2 - \frac{n-1}{2}\gamma^1\beta_{12} - \frac{n+1}{2}\gamma^3\beta_{23} + \frac{2n-1}{6}\gamma^1\gamma^3\beta_{123} \right) \\
\mathcal{G}_{\alpha_2+\alpha_3; n+1/2} &= (\gamma^1)^n \left( (n+1)\beta_{23} - \frac{n-1}{2}\gamma^1\beta_{123} \right)
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
\overline{\mathcal{G}}_{-\alpha_2; n-1/2} &= -(\gamma^1)^n \left( \frac{1}{2} \gamma^1 \gamma^2 + \gamma^{12} \right) \beta_1 - n(\gamma^1)^{n-1} \gamma^2 \gamma^{12} \beta_2 \\
&- (\gamma^1)^{n-1} \left( \frac{2n-3}{6} \gamma^1 \gamma^2 \gamma^3 - \frac{n-2}{2} \gamma^1 \gamma^{23} + \frac{1}{2} n \gamma^{12} \gamma^3 - n \gamma^{123} \right) \beta_3 \\
&+ \frac{n-1}{2} (\gamma^1)^n \gamma^2 \gamma^{12} \beta_{12} \\
&- (\gamma^1)^{n-2} \left( \frac{n^2-2}{4} (\gamma^1)^2 \gamma^2 \gamma^{23} + \frac{n(n+2)}{12} \gamma^1 \gamma^2 \gamma^3 \gamma^{12} \right. \\
&\quad \left. + \frac{n(n-1)}{2} \gamma^1 \gamma^{12} \gamma^{23} + \frac{n^2}{2} \gamma^1 \gamma^2 \gamma^{123} + n(n-1) \gamma^{12} \gamma^{123} \right) \beta_{23} \\
&+ (\gamma^1)^{n-1} \left( \frac{n^2-4}{24} \gamma^1 \gamma^2 \gamma^{12} \gamma^3 + \frac{3n^2-4n-4}{24} (\gamma^1)^2 \gamma^2 \gamma^{23} \right. \\
&\quad \left. + \frac{n(n-2)}{4} \gamma^1 \gamma^{12} \gamma^{23} + \frac{(3n-4)n}{12} \gamma^1 \gamma^2 \gamma^{123} + \frac{n(n-2)}{2} \gamma^{12} \gamma^{123} \right) \beta_{123} \\
&+ n(\gamma^1)^{n-1} \left( \frac{1}{2} \gamma^1 \gamma^2 + \gamma^{12} \right) \sqrt{k} \partial \varphi_1 \\
&+ (\gamma^1)^{n-1} \left( \frac{n-2}{2} \gamma^1 \gamma^2 + n \gamma^{12} \right) \sqrt{k} \partial \varphi_2 \\
&+ \frac{(6kn-3n-8)n}{12} (\gamma^1)^{n-1} \gamma^2 \partial \gamma^1 + (k-1/2)(n^2-n)(\gamma^1)^{n-2} \gamma^{12} \partial \gamma^1 \\
&+ \frac{(3n-6)k-n}{6} (\gamma^1)^n \partial \gamma^2 + (k-1/2)n(\gamma^1)^{n-1} \partial \gamma^{12} \\
\overline{\mathcal{G}}_{-(\alpha_2+\alpha_3); n-1/2} &= -(\gamma^1)^n \left( \frac{1}{6} \gamma^1 \gamma^2 \gamma^3 + \frac{1}{2} \gamma^1 \gamma^{23} + \frac{1}{2} \gamma^{12} \gamma^3 + \gamma^{123} \right) \beta_1 \\
&+ (\gamma^1)^{n-2} \left( \frac{n(n-7)}{12} \gamma^1 \gamma^2 \gamma^{12} \gamma^3 + \frac{n^2+n-4}{4} (\gamma^1)^2 \gamma^2 \gamma^{23} \right. \\
&\quad \left. + \frac{n(n+1)}{2} \gamma^1 \gamma^{12} \gamma^{23} + \frac{n(n-1)}{2} \gamma^1 \gamma^2 \gamma^{123} + n(n-1) \gamma^{12} \gamma^{123} \right) \beta_2 \\
&- (\gamma^1)^{n-1} \gamma^3 \left( \frac{2n-3}{6} \gamma^1 \gamma^2 \gamma^3 - \frac{n-2}{2} \gamma^1 \gamma^{23} + \frac{n}{2} \gamma^{12} \gamma^3 - n \gamma^{123} \right) \beta_3 \\
&- (\gamma^1)^{n-1} \left( \frac{n^2-9n+6}{24} \gamma^1 \gamma^2 \gamma^{12} \gamma^3 + \frac{n^2-n-2}{8} (\gamma^1)^2 \gamma^2 \gamma^{23} \right. \\
&\quad \left. + \frac{n(n-1)}{4} \gamma^1 \gamma^{12} \gamma^{23} + \frac{n^2-3n+2}{4} \gamma^1 \gamma^2 \gamma^{123} + \frac{n(n-3)}{2} \gamma^{12} \gamma^{123} \right) \beta_{12} \\
&- (\gamma^1)^{n-2} \left( \frac{n(n+5)}{24} \gamma^1 \gamma^2 \gamma^{12} (\gamma^3)^2 + \frac{n(n-1)}{4} \gamma^1 \gamma^{12} \gamma^{23} \gamma^3 \right. \\
&\quad \left. + \frac{n(n-1)}{4} \gamma^1 \gamma^2 \gamma^3 \gamma^{123} + \frac{(3n-1)n}{24} (\gamma^1)^2 \gamma^2 \gamma^{23} \gamma^3 \right. \\
&\quad \left. + \frac{n(n-1)}{2} \gamma^{12} \gamma^3 \gamma^{123} + n \gamma^1 \gamma^{23} \gamma^{123} \right) \beta_{23}
\end{aligned}$$

$$\begin{aligned}
& + (\gamma^1)^{n-1} \left( \frac{2n^2 + 7n - 15}{72} \gamma^1 \gamma^2 \gamma^{12} (\gamma^3)^2 + \frac{2n^2 - n - 3}{12} \gamma^1 \gamma^{12} \gamma^{23} \gamma^3 \right. \\
& + \frac{2n^2 - 3n + 1}{12} \gamma^1 \gamma^2 \gamma^3 \gamma^{123} + \frac{2n^2 - n - 3}{24} (\gamma^1)^2 \gamma^2 \gamma^{23} \gamma^3 \\
& \left. + \frac{n(n-1)}{3} \gamma^{12} \gamma^3 \gamma^{123} + \frac{n-1}{2} \gamma^1 \gamma^{23} \gamma^{123} \right) \beta_{123} \\
& + (\gamma^1)^{n-1} \left( \frac{n}{6} \gamma^1 \gamma^2 \gamma^3 + \frac{n}{2} \gamma^1 \gamma^{23} + \frac{n}{2} \gamma^{12} \gamma^3 + n \gamma^{123} \right) \sqrt{k} \partial \varphi_1 \\
& + (\gamma^1)^{n-1} \left( \frac{n-3}{6} \gamma^1 \gamma^2 \gamma^3 + \frac{n-2}{2} \gamma^1 \gamma^{23} + \frac{n}{2} \gamma^{12} \gamma^3 + n \gamma^{123} \right) \sqrt{k} \partial \varphi_2 \\
& + (\gamma^1)^{n-1} \left( \frac{2n-3}{6} \gamma^1 \gamma^2 \gamma^3 - \frac{n-2}{2} \gamma^1 \gamma^{23} + \frac{n}{2} \gamma^{12} \gamma^3 - n \gamma^{123} \right) \sqrt{k} \partial \varphi_3 \\
& + (\gamma^1)^{n-2} \left( \frac{(4kn - 3n - 7)n}{24} \gamma^1 \gamma^2 \gamma^3 + \frac{(3k-2)n(n-1)}{6} \gamma^{12} \gamma^3 \right. \\
& \left. + \frac{(2kn - n - 3)n}{4} \gamma^1 \gamma^{23} + (k-1/2)n(n-1) \gamma^{123} \right) \partial \gamma^1 \\
& + \frac{kn - n + 1}{6} (\gamma^1)^n \gamma^3 \partial \gamma^2 + \frac{(k-1)n}{2} (\gamma^1)^{n-1} \gamma^3 \partial \gamma^{12} \\
& - (\gamma^1)^{n-1} \left( \frac{8kn - 12k + n^2 + 11n - 22}{24} \gamma^1 \gamma^2 + \frac{(6k + n + 11)n}{12} \gamma^{12} \right) \partial \gamma^3 \\
& + \frac{kn - 2k - 1}{2} (\gamma^1)^n \partial \gamma^{23} + kn (\gamma^1)^{n-1} \partial \gamma^{123} \tag{50}
\end{aligned}$$

The affine  $SL(2)$  current subalgebra is realized as

$$\begin{aligned}
\tilde{\mathcal{E}}_n &= -(\gamma^1)^n \beta_3 - (\gamma^1)^{n-1} \left( \frac{n+1}{2} \gamma^1 \gamma^2 + n \gamma^{12} \right) \beta_{23} \\
&+ (\gamma^1)^n \left( \frac{3n-1}{12} \gamma^1 \gamma^2 + \frac{n-1}{2} \gamma^{12} \right) \beta_{123} \\
\tilde{\mathcal{H}}_n &= (\gamma^1)^{n-1} \left( \frac{n+2}{2} \gamma^1 \gamma^2 + n \gamma^{12} \right) \beta_2 - 2(\gamma^1)^n \gamma^3 \beta_3 \\
&- (\gamma^1)^n \left( \frac{n}{4} \gamma^1 \gamma^2 + \frac{n-2}{2} \gamma^{12} \right) \beta_{12} \\
&- (\gamma^1)^{n-1} \left( \frac{5n}{12} \gamma^1 \gamma^2 \gamma^3 + \frac{n+2}{2} \gamma^1 \gamma^{23} + n \gamma^{12} \gamma^3 + n \gamma^{123} \right) \beta_{23} \\
&+ (\gamma^1)^n \left( \frac{n}{4} \gamma^1 \gamma^2 \gamma^3 + \frac{n}{4} \gamma^1 \gamma^{23} + \frac{7n}{12} \gamma^{12} \gamma^3 + \frac{n-2}{2} \gamma^{123} \right) \beta_{123} \\
&+ (\gamma^1)^n \sqrt{k} \partial \varphi_3 \\
\tilde{\mathcal{F}}_n &= -(\gamma^1)^{n-1} \left( \frac{n+3}{6} \gamma^1 \gamma^2 \gamma^3 + \frac{n+2}{2} \gamma^1 \gamma^{23} + \frac{n}{2} \gamma^{12} \gamma^3 + n \gamma^{123} \right) \beta_2 \\
&+ (\gamma^1)^n (\gamma^3)^2 \beta_3 \\
&+ (\gamma^1)^n \left( \frac{n+1}{12} \gamma^1 \gamma^2 \gamma^3 + \frac{n}{4} \gamma^1 \gamma^{23} + \frac{n-2}{4} \gamma^{12} \gamma^3 + \frac{n-2}{2} \gamma^{123} \right) \beta_{12} \\
&+ (\gamma^1)^{n-1} \gamma^3 \left( \frac{n-3}{12} \gamma^1 \gamma^2 \gamma^3 + \frac{n+2}{4} \gamma^1 \gamma^{23} + \frac{n}{4} \gamma^{12} \gamma^3 + \frac{n}{2} \gamma^{123} \right) \beta_{23}
\end{aligned}$$



$$\begin{aligned}
& - (\gamma^1)^n \gamma^3 \left( \frac{n}{18} \gamma^1 \gamma^2 \gamma^3 + \frac{2n+1}{12} \gamma^1 \gamma^{23} + \frac{2n+3}{12} \gamma^{12} \gamma^3 + \frac{2n-3}{6} \gamma^{123} \right) \beta_{123} \\
& - (\gamma^1)^n \gamma^3 \sqrt{k} \partial \varphi_3 + \left( k + \frac{n}{12} + 1 \right) (\gamma^1)^n \partial \gamma^3
\end{aligned} \tag{51}$$

whereas the  $U(1)$  generator is

$$U_n = \oint \frac{dz}{2\pi i} (\gamma^1)^n \sqrt{k} (-\partial \varphi_1 - 2\partial \varphi_2 + \partial \varphi_3) \tag{52}$$

The remaining 2 fermionic spin 1/2 generators and the bosonic scalar are

$$\begin{aligned}
\phi_{-\alpha_2; n-1/2} &= \oint \frac{dz}{2\pi i} (\gamma^1)^{n-1} V_{-\alpha_2}^{\alpha_1} \sqrt{k} (-\partial \varphi_1 - 2\partial \varphi_2 + \partial \varphi_3) \\
\phi_{-(\alpha_2+\alpha_3); n-1/2} &= \oint \frac{dz}{2\pi i} (\gamma^1)^{n-1} V_{-(\alpha_2+\alpha_3)}^{\alpha_1} \sqrt{k} (-\partial \varphi_1 - 2\partial \varphi_2 + \partial \varphi_3)
\end{aligned} \tag{53}$$

and

$$S_n = \oint \frac{dz}{2\pi i} (\gamma^1)^{n-1} V_{-(\alpha_2+\alpha_3)}^{\alpha_1} V_{-\alpha_2}^{\alpha_1} \sqrt{k} (-\partial \varphi_1 - 2\partial \varphi_2 + \partial \varphi_3) \tag{54}$$

According to (46), the polynomials  $V_{-\alpha_2}^{\alpha_1}$  and  $V_{-(\alpha_2+\alpha_3)}^{\alpha_1}$  are

$$\begin{aligned}
V_{-\alpha_2}^{\alpha_1} &= \frac{1}{2} \gamma^1 \gamma^2 + \gamma^{12} \\
V_{-(\alpha_2+\alpha_3)}^{\alpha_1} &= \frac{1}{6} \gamma^1 \gamma^2 \gamma^3 + \frac{1}{2} \gamma^1 \gamma^{23} + \frac{1}{2} \gamma^{12} \gamma^3 + \gamma^{123}
\end{aligned} \tag{55}$$

## References

- [1] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231.
- [2] S. Gubser, I. Klebanov and A. Polyakov, Phys. Lett. **B 428** (1998) 105.
- [3] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253.
- [4] J. Rasmussen, *Constructing Classical and Quantum Superconformal Algebras on the Boundary of  $AdS_3$* , hep-th/0002188.
- [5] K. Ito, Phys. Lett. **B 449** (1999) 48.
- [6] A. Giveon, D. Kutasov and N. Seiberg, Adv. Theor. Math. Phys. **2** (1998) 733.
- [7] O. Andreev, Nucl. Phys. **B 552** (1999) 169; *Unitary Representations of some Infinite Dimensional Lie Algebras Motivated by String Theory on  $AdS_3$* , hep-th/9905002.
- [8] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, Phys. Lett. **B 62** (1976) 105; Nucl. Phys. **B 114** (1976) 297.
- [9] K. Schoutens, Phys. Lett. **B 194** (1987) 75; Nucl. Phys. **B 295** (1988) 634.

- [10] A. Sevrin, W. Troost and A. Van Proyen, Phys. Lett. **B 208** (1988) 447.
- [11] A. Ali and A. Kumar, Mod. Phys. Lett. **A 8** (1993) 1527.
- [12] A. Ali, *Classification of Two Dimensional  $N = 4$  Superconformal Symmetries*, hep-th/9906096.
- [13] J. Rasmussen, Nucl. Phys. **B 510** (1998) 688.
- [14] K. Ito, J.O. Madsen and J.L. Petersen, Nucl. Phys. **B 398** (1993) 425.
- [15] D.-S. Tang, J. Math. Phys. **25** (1984) 2966.